

## Special Session 51: Ordinal Symbolic Dynamics and Applications

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Ordinal symbolic dynamics is a promising new approach to the investigation of time series and the systems behind them, with the important properties of being conceptually simple and relatively robust with respect to noise. The idea behind it is to consider the order relation between the values of a time series instead of the values themselves. Roughly speaking, a given time series is transformed into a series of order patterns describing the ups and downs of the original series. Then the distribution of ordinal patterns obtained is the basis of the analysis.

Since Bandt and Pompe introduced permutation entropy in their celebrated paper in 2002, the idea has been intensively discussed. For example, there are interesting results and open questions concerning the relation of permutation entropy to conventional ergodic concepts in dynamical systems like Kolmogorov-Sinai entropy. On the other hand, the tools of ordinal symbolic dynamics are being used in real time series analysis with a remarkable success, e.g. for identifying and discriminating different brain states in epilepsy research and in anesthesiology, for heart rate analysis, and for testing independence.

The purpose of the special session is to bring together researchers working on ordinal symbolic dynamics and related topics, in order to discuss new developments in theory and applications, and to celebrate the first ten years of the field!

### Permutation entropy: one concept, two approaches

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The original definition of permutation entropy by C. Bandt and B. Pompe has the theoretical appeal of involving only one infinite process (a limit over the length of the ordinal patterns), partitions being chosen in a canonical way. The relation between the Kolmogorov-Sinai entropy and the permutation entropy for measurable selfmaps of  $R^m$  has been recently studied in depth by K. Keller.

Alternatively, one could mimic the Kolmogorov approach to the conventional metric entropy and use arbitrary partitions instead. As compared to the former approach, the latter involves an additional infinite process, namely, a limit over finer and finer partitions. Along these lines one can introduce permutation entropy even if the state space is not linearly ordered. Not unexpectedly, in this second approach permutation entropy can be shown to coincide with the Kolmogorov-Sinai entropy. In this regard, let us also mention the recent results of T. Haruna and K. Nakajima concerning ordinal symbolic dynamics with a finite alphabet.

Both approaches have their theoretical and practical virtues. In fact, the theoretical results obtained along these two ways are very similar, though not the same. Under which assumptions they coincide, remains an interesting and difficult open problem.

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### Event detection, multimodality and non-stationarity: order patterns, a tool to rule them all?

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The control of complex systems involving non-stationary dynamics require very fast and precise methods to identify and correlate events. These methods should combine flexibility and robustness: perfect estimation of all states cannot be performed, but it is necessary to infer enough information to construct efficient actuation laws. Moreover, the presence of noise should be treated conveniently. In this concern, fine-grained orbits could contain too much information to derive an adequate control strategy and, consequently, it could be advisable to discretize them by assigning a proper partition to the phase space. As a matter of fact, it is possible to reconstruct the underlying dynamics of time series upon Markov partitions (or at least generating ones). From a theoretical point of view, the existence of generating partitions is guaranteed for any ergodic process. Nevertheless, the practical counterpart of this result cannot be embodied straightforward. Certainly, different methods have been proposed to obtain the generating partition of a given dynamical system, but this problem is still an open and hard challenge in applied dynamical systems theory. On the other hand, symbolic dynamics is a very useful tool when considering different sources of information, which is the case of multimodal systems. The discretization of time series allows to integrate and compare different dynamics independently of their original codification and accordingly to the basis of information theory.

Order patterns are an alternative to get an approximation of Markov partitions, or at least of gen-

erating ones. The seminal work of Bandt and Pompe in 2002 showed that entropy can be estimated using the ordinal time series of dynamical systems. The so-called permutation entropy can be computed through the histogram of order patterns, which enables the classification of different dynamics and the detection of changes in them. In this vein, the analysis of order patterns has been successfully applied to the uni/multivariate study of EEG signals and Event Related Potentials (ERPs), to the detection of dynamical changes, to the estimation of control parameters, and to the characterization of synchronization and couplings. However, most of previous works on order patterns rely on the offline analysis of long time series. In this work we study the limitations of asymptotic analysis of intra and extra-cellular time series by means of the corresponding order patterns. Specifically we derive different measures from ordinal time series in order to discuss their suitability when real-time estimation and actuation is considered. Additionally, different connectivity measures between ordinal time series are evaluated to establish their adequacy to model interactions in a multimodal setup.

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#### Iteration of differentiable functions under m-modal maps with aperiodic kneading sequences

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We consider an infinite dynamical system obtained by iteration of functions of a class of differentiable functions,  $A$ , under  $m$ -modal maps  $f$ . Using an algorithm, we obtained some numerical and symbolic results related to the frequencies of occurrence of critical values of the iterated functions when the kneading sequences of  $f$  are aperiodic. Moreover, we analyze the evolution as well as the distribution of the aperiodic critical values of the iterated functions.

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#### Analysis of heart rate asymmetry by ordinal patterns

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The problem of recognition of all main features of healthy heart rhythm (so-called sinus rhythm) is still one of the biggest challenges in contemporary cardiology. Recently, there has been observed an

interesting physiological phenomenon of heart rate asymmetry related to unequal contribution of heart rate decelerations and accelerations to heart rate variability. We apply ordinal patterns and permutation entropy to the analysis of ECG time series (RR intervals) and observe further new features of heart rhythm asymmetry. Ordinal patterns analysis might be successfully used for heart rate description in healthy subjects and could provide a reference for identification of pathological states.

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#### Permutation approach to finite-alphabet stationary stochastic processes based on the duality between values and orderings

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The duality between values and orderings is a powerful method to establish equalities between various information theoretic measures and their permutation versions for finite-alphabet stationary stochastic processes. To illustrate its potency, in this talk, we consider the transfer entropy (TE) which is a measure of the direction and magnitude of information flow between two jointly distributed processes. There are two known permutation versions of TE: the symbolic transfer entropy (STE) and the transfer entropy on rank vectors (TERV). We show that the rate versions of both STE and TERV are equal to TE rate for any finite-state finite-alphabet hidden Markov model with an ergodic internal process.

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#### KS entropy and permutation entropy

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The relationship of KS entropy and Permutation entropy in its original version given by Bandt and Pompe is an interesting open problem. Whereas both entropies are equal for piecewise monotone interval maps as shown by Bandt, Pompe and G. Keller, there is nothing known about coincidence of the entropies for other classes of dynamical systems.

In our talk, we discuss the problem for a generalized concept of permutation entropy, which is based on considering observables on a (multi-dimensional) dynamical system. In particular, we show that KS entropy cannot be larger than Permutation entropy and characterize equality of KS entropy and Permutation entropy on the pure ordinal level. We include Amigo's version of Permutation entropy, which is equivalent to KS entropy, into the discussion.

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### Cardiovascular regulation during sleep quantified by symbolic coupling traces

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Sleep is a complex regulated process with short periods of wakefulness and different sleep stages. These sleep stages modulate autonomous functions such as blood pressure, and heart rate. The method of symbolic coupling traces (SCT) is used to analyze and quantify time-delayed coupling of these measurements during different sleep stages. The symbolic coupling traces, defined as the symmetric and diametric traces of the bivariate word distribution matrix, allow for the quantification of time-delayed coupling. In this paper, the method is applied to heart rate and systolic blood pressure time series during different sleep stages for healthy controls as well as for normo- and hypertensive patients with sleep apneas. Using the SCT, significant different cardiovascular mechanisms not only between the deep sleep and the other sleep stages but also between healthy subjects and patients can be revealed. The SCT method is applied to model systems, compared with established methods, such as cross correlation, mutual information and cross recurrence analysis and demonstrates its advantages especially for non-stationary physiological data. Thereby SCT proves to be more specific in detecting delays of directional interactions than standard coupling analysis methods and yields additional information which can not be measured by standard parameters of heart rate- and blood pressure variability. The proposed method may help to indicate pathological changes in cardiovascular regulation and also effects of continuous positive airway pressure therapy on the cardiovascular system.

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### Symbolic analysis: inference basis for constructing hypotheses

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This paper introduces a novel method for elaborating nonparametric hypothesis tests based on symbolic analysis. The existing tests based on symbolic entropy that have been used for testing central hypotheses in economics, finance and spatial econometrics are now particular cases of this general approach. This family of potential symbolic tests benefits from the use of few assumptions that limit the general applicability of any test. In addition, as a result of the method, we easily construct and provide four new statistics for testing the null of spatiotemporal

independence. Monte Carlo results highlight the well behavior of the proposed test.

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### Information measures to characterize the coupling complexity between dynamical system components

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The concept of transcripts arises naturally when studying relationships between coupled dynamical systems using ordinal symbolic dynamics. Since transcripts belong to the family of ordinal patterns, they also exhibit properties like robustness to noise, computational speed, and a sound theoretical framework. In contrast to other methods of ordinal symbolic dynamics, transcripts explicitly exploit the group theoretical properties of permutations. Using this approach, we propose information measures suitable to characterize the complexity between components of a dynamical systems. The so-called "Coupling Complexity" differs in general from the complexity of the complete system or its individual components. We apply and test the performance of these information measures with numerical and real world data.

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### Local information dynamics via permutation-information theoretic approach

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Taichi Haruna

Recently, permutation-information theoretic approach has been used in a broad range of research field. In particular, in a high-dimensional dynamical systems research, it has been shown that this approach can be effectively used to characterize a global property including complexity of their spatiotemporal dynamics. Here, we show that the approach can be also applied to reveal local informational dynamics existing in high-dimensional dynamical systems. J. T. Lizier et al. have recently introduced the concept of local information dynamics, which consists of information storage, transfer, and modification. This concept has been intensively studied with regard to cellular automata and has provided quantitative evidence of several characteristic behaviors observed in the system. In this study, we demonstrate that the application of the permutation-information theoretic approach, which introduces natural symbolization

methods, makes the concept easy to extend to systems that have continuous states. We then apply this approach to coupled map lattices, demonstrating that it can be successfully used as a spatiotemporal filter to stress a coherent structure hidden in the system. In particular, we show that the approach can clearly stress out defect turbulences or Brownian motion of defects from the background. Finally, we demonstrate and discuss scenarios for the application of this approach within the robotics community.

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### Nonlinear signal analysis and classification using ordinal patterns

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Ordinal patterns describing amplitude relations in (short) segments of time series have successfully been used in a wide range of applications like detection of determinism in noisy time series, estimation of transfer entropy in epilepsy, or complexity analysis of time series. In this contribution we shall present and discuss applications of ordinal pattern statistics for nonlinear (deterministic) chaotic systems, including synchronization analysis, forecasting, and signal classification. In particular promising applications to ECG data will be presented which show that symbolic dynamics based on ordinal patterns provides a powerful tool for coping with data from life sciences.

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### LE–statistic: a versatile tool in ordinal time series analysis

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We introduce a quantity called *LE–statistic*. It is an easily computable functional of two ordinal time series with versatile applications. We demonstrate its usefulness as (i) a statistic in a nonparametric dependence test of two time series, and (ii) a complexity measure of one time series. For chaotic orbits of one–dimensional systems it is related to the Lyapunov characteristic exponent resp. metric entropy.

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### Ordinal pattern distributions

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The distribution of ordinal patterns describing the up and down in the dynamics of a system contains much information on the structure of the system. This fact is illustrated by some examples from dynamical systems and stochastic processes. On one hand we show how simple measures based on the ordinal pattern distribution can be used for the estimation of model parameters, and on the other hand we discuss whether deterministic and stochastic behavior can be distinguished by looking at the ordinal pattern distributions obtained from a system.

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### Conditional entropy of ordinal patterns

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Ordinal patterns of order  $d$  reflect order relations between  $(d + 1)$  successive points of an orbit of a dynamical system. In our talk, we introduce a conditional entropy measuring the uncertainty of  $(n + 1)$  successive ordinal patterns of some order  $d$  given the first  $n$  ordinal patterns. The use of such conditional entropy provides interesting insights into the problem of the relationship of KS entropy and Permutation entropy. In particular, we provide results that relate these two quantities to conditional entropy of ordinal patterns in some special cases.

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