# Special Session 11: Advances in Classical and Geophysical Fluid Dynamics

**Madalina Petcu**, Univ Poitiers, France
**Roger Temam**, Indiana University, USA
**Shouhong Wang**, Indiana University, USA

Geophysical Fluid Dynamics (GFD) came of age in the 1950s and 1960s, with the mathematically inspired work of Charney, Lorenz, Stommel, Veronis and von Neumann, among others. In the last twenty years or so, mathematical, statistical and numerical developments for the Navier-Stokes equations and basic geophysical fluid dynamics models are stimulating a deepening and broadening of its central concerns. The aim of this special session is to bring together a number of researchers who work on diverse frontiers of GFD and Navier-Stokes equations and their applications to climate dynamics and environmental problems to discuss how new ideas and methods will advance the field in the next decade.

### A model equation for water waves with dissipation

**Jerry Bona**
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**David Ambrose, David Nicholls**

The well-posedness of a model for water waves with dissipation put forward by Diaz, Dyachenko and Zhakharov is considered. Some associated ill-posedness results will also be outlined.

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### Invariant measures for dissipative dynamical systems: abstract results and applications

**Mickael Chekroun**
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**Nathan Glatt-Holtz**

In this talk we will study certain invariant measures that can be associated to the time averaged observation of a broad class of dissipative semigroups via the notion of a generalized Banach limit. Consider an arbitrary complete separable metric space $X$ which is acted on by any continuous semigroup $\{S(t)\}_{t\geq 0}$. Suppose that $\{S(t)\}_{t\geq 0}$ possesses a global attractor $\mathcal{A}$. We will show that, for any generalized Banach limit $\underset{T\to\infty}{\mathrm{LIM}}$ and any distribution of initial conditions $\mathfrak{m}_0$, that there exists an invariant probability measure $\mathfrak{m}$, whose support is contained in $\mathcal{A}$, such that

$$\int_X \phi(x)d\mathfrak{m}(x) = \underset{T\to\infty}{\mathrm{LIM}} \frac{1}{T}\int_0^T \int_X \phi(S(t)x)d\mathfrak{m}_0(x)dt,$$

for all observables $\phi$ living in a suitable function space of continuous mappings on $X$.This work is based on a functional analytic framework simplifying and generalizing previous works in this direction.

Two examples of concrete dynamical systems where the semigroup is known to be non-compact will be then examined in detail. We first consider the Navier-Stokes equations with memory in the diffusion terms. This is the so called Jeffery's model which describes certain classes of viscoelastic fluids. We then consider a family of neutral delay differential equations, that is equations with delays in the time derivative terms. These systems may arise in the study of wave propagation problems coming from certain first order hyperbolic partial differential equations; for example for the study of line transmission problems.For the second example the phase space is $X = C([-\tau, 0], \mathbb{R}^n)$, for some delay $\tau > 0$, so that $X$ is not reflexive in this case.

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### Singular limits of geophysical fluid dynamics in spherical and bounded domains.

**Bin Cheng**
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We study 2D geophysical fluids that are inviscid and under strong Coriolis force. In particular, we address the challenges of nontrivial geometry of the physical domains such as on a sphere. We also consider general (a.k.a ill-prepared) initial data. The first model is incompressible Euler equations on a fast rotating sphere. Motivated by recent studies in geophysical and planetary sciences, we investigate the finite-time-average of the solution and prove that it stays close to a subspace of longitude-independent zonal flows. The second model is the rotating shallow water equations with solid-wall boundary conditions. We show that in the zero-Rossby-number limit, the solution comprises a quasi-geophysical flow and fast gravity waves, both subject to the same solid-wall boundary conditions.

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### Qualitative properties of eigenvectors related to multivoque operators

**Houssam Chrayteh**
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We introduce the notion of $\overrightarrow{\beta}$-multivoque Leray-Lions operators that are strongly monotonic on a Banach-Sobolev function space $V$ and we study the generalized eigenvalue problem $Au = \lambda\partial j(u)$. Here $\partial j$ denotes the subdifferential in the sense of convex analysis or more generally in the sense of H. Clarke. Connected with this problem, we also study a minimization problem with constraint then we give some qualitative properties of solutions by using relative rearragement.

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**New results for the stochastic PDEs of fluid dynamics**

**Nathan Glatt-Holtz**
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The addition of white noise driven terms to the fundamental equations of physics and engineering are used to model numerical and empirical uncertainties. In this talk we will discuss some recent results for the Stochastic Navier-Stokes and Euler Equations as well as for the Stochastic Primitive Equations, a basic model in geophysical scale fluid flows. For all of the above systems our results cover the case of a general nonlinear multiplicative stochastic forcing.

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**Viscous asymptotic models for water waves**

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In this talk, we discuss some issues related to various viscous asymptotic models for water waves. These models range from Kakutani-Maatsuchi models to models derived a decade ago independently by Dutykh and Dias, and Liu et Orfila. We focus on the decay rate of solutions to these models.

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**Time periodic solutions of the Primitive equations of the large-scale ocean**

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**Ming-Cheng Shiue**

This is a joint work with Ming-Cheng Shiue. For several decades, concerning the long time behavior of fluid motion, the time periodic flows have become an important type of flow patterns. In 1959, Serrin proposed a very heuristic method for proving the existence of asymptotic stable periodic solutions of the Navier-Stokes equations with small periodic forcing terms under suitable assumptions. Namely, in such case, one may prove that every small ( in a suitable sense ) solution would converge to a time periodic solution ( with the same period as the non-trivial forcing term ). In this article, we prove the existence of time periodic solution for the 3-D primitive equation with suitable time periodic forcing condition. Some related asymptotic behaviors of the solutions are also demonstrated.

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**The linearized 2D inviscid shallow water equations in a rectangle: boundary conditions and well-posedness**

**Aimin Huang**
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**Roger Temam**

We consider the linearized 2D inviscid shallow water equations in a rectangle. A set of boundary conditions is proposed which make these equations well-posed. Several different cases occur depending on the relative values of the reference velocities $(u_0, v_0)$ and reference height $\phi_0$ (sub- or super-critical flow at each part of the boundary).

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**Center manifold reduction for stochastic partial differential equations**

**Honghu Liu**
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**Mickael D. Chekroun, Shouhong Wang**

We present – within the framework of random dynamical systems – explicit approximation formulas for center manifold reduction of a broad class of nonlinear stochastic evolution equations with linear multiplicative white noise. This reduction procedure allows us to derive explicit reduced equations to the corresponding random center manifold and conduct detailed analysis on the long term dynamics and bifurcation associated with both the reduced and the original equations. As an application, we analyze the bifurcation scenarios of the reduced equation of the Rayleigh-Benard convection with linear multiplicative noise. This is a joint work with Mickael D. Chekroun and Shouhong Wang.

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**Some equations with logarithmic nonlinear terms**

**Alain Miranville**
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Our aim in this talk is to discuss the well-posedness and the asymptotic behavior, in terms of finite-dimensional attractors, of several equations with logarithmic nonlinear terms. Such equations arise, e.g., in phase transition.

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**The two layers shallow water equations**

**Madalina Petcu**
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The aim of this talk is to model the flow of two superposed layers of fluid governed by the shallow water equations in space dimension one. Under some suitable hypotheses the governing equations are hyperbolic. We propose suitable boundary conditions and establish a result of existence and uniqueness of smooth solutions for a limited time for our model.

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**Lower bounds on blow-up solutions of the 3D Navier-Stokes equations**

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**Ricardo Perreira da Silva & Witold Sadowski**

If $u(t)$ is a solution of the 3D Navier-Stokes equations with a regular initial condition $u_0 \in H^s$ with $s \geq 1/2$, it is know that $u(t)$ remains bounded in $H^s$ for some short time interval.If one assumes that the solution "blows up" at some time $t = T$, then relatively simple arguments imply that $\|u(T-t)\|_{H^1} \geq ct^{-1/4}$, and such a blow up rate is consistent with the natural scaling of the Navier-Stokes equations on the whole space.In this talk I will discuss what can be said about the blow up rate of solutions in other Sobolev spaces $H^s$ for $s \geq 1/2$.

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**Quasi-hydrostatic modelling in geophysical fluid dynamics**

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This works deals with the derivation of the quasi-hydrostatic quasi-geostrophic (QHQG) equations of large scale ocean as the Rossby number goes to zero. We follow classical techniques for the derivation of the quasi-geostrophic (QG) equations but the primitive equations that we consider account for all the rotating terms. In particular, the traditional approximation on the Coriolis force (and consequently the hydrostatic approximation) are not made in the vertical momentum equation, which reduces to

$$-f^*u + \frac{\partial \varphi}{\partial z} = -\frac{g\,\varrho}{\varrho_0}.$$

We end up with a slightly different QG model:

$$D[q + \beta y] = 0, \quad (3)$$

$$\Big(\Delta + \partial_Z[\frac{1}{N^2}\partial_Z]\Big)\phi = q, \quad (4)$$

where $D = \partial_t + J[\phi,\ ]$, $\Delta = \partial_x^2 + \partial_y^2$, $J[a,b] = a_x b_y - a_y b_x$, $\partial_Z = \partial_z + \lambda\partial_y$, and $\lambda = \frac{H}{L}\cot\theta_0$.

The parameter $\lambda$ above measures the non-traditionnal Coriolis effects ($\lambda = 0$ in the traditional QG model). It is actually responsible for a (slight) tilt of the vertical direction ($\partial_Z = \partial_z + \lambda\partial_y$), which has been illustrated in previous works using the primitive equations. This work can be seen as a first mathematical illustration of these papers.

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**A numerical study of the boundary value problem for the shallow water equations**

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In geophysical fluid dynamics, Limited Area Models have been used to achieve high resolution over a region of interest. However, this forces us to face the choice of boundary conditions since there is no physical law which can provide natural boundary conditions at the lateral boundary. In this talk, one-layer and two-layer shallow water equations related to weather prediction and oceanography are considered. We will discuss the suitability of the proposed boundary conditions and the suitability of the proposed numerical schemes which are inspired by the semidiscrete central-upwind method.

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**Near-equilibrium behavior of 2-D stochastic Navier-Stokes equations**

**Eric Simonnet**
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We present recent results on the near-equilibrium behavior of 2-D stochastic Navier-Stokesequations in doubly-periodic and channel flow domains and discuss its link withthe statistical mechanics of 2-D Euler equation obtained by Robert, Sommeria and Miller in the 90s.We also discuss numerical algorithms for estimating extreme events and large-deviation rate functionsin the context of PDEs.

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**Pullback attractors for the 2D Primitive equations of the ocean.**

**Theodore Tachim Medjo**
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We derive a sufficient condition for the existence of a pullback attractor for closed cocycles acting on a Banach space, where the strong continuity is replaced by a much weaker requirement that the cocycle be a closed map. The proof of the existence of the attractor uses the concept of measure of non-compactness and a result from V. Pata and S. V. Zelik (2007). The result extends that obtained by H. Song and H. Q. Wu (2007). As application, we prove the existence of the pullback attractor of a cocycle associated with the z-strong solutions of a non-autonomous two-dimensional primitive equations of the ocean.

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**The Zakharov-Kuznetsov equation of plasma physics: the case of a bounded domain**

**Roger Temam**
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**Jean-Claude Saut, Chuntian Wang**

In this lecture we will present some recent results on existence of solutions for the Zakharov-Kuznetsov equation of plasma physics (space dimension two and three). In space dimension two the solution constructed is unique, and more regular solutions can be also constructed. The case considered here is the case of a bounded domain which raises many new difficulties compared to the case of a flow in an infinite strip considered earlier.

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**Fluctuation-dissipation theory with time periodic forcing**

**Xiaoming Wang**
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**Andrew Majda**

We present a new fluctuation-dissipation theory (linear response formulas) for chaotic (stochastic) dynamical systems with time periodic coefficients. Such time periodic systems arise naturally in climate change studies due to the seasonal cycle. These response formulas are developed through the mathematical interplay between statistical solutions for the time-periodic dynamical systems and the related skew-product system. This interplay is utilized to develop new systematic quasi-Gaussian and adjoint algorithms for calculating the climate response in such time-periodic systems. New linear response formulas are also developed here for general time-dependent statistical ensembles arising in ensemble prediction including the effects of deterministic model errors, initial ensembles, and model noise perturbations simultaneously . An information theoretic perspective is developed in calculating those model perturbations which yield the largest information deficit for the unperturbed system both for climate response and finite ensemble predictions. This is a joint work with Andy Majda.

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**Dynamic transition theory for thermohaline circulation**

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The main objective of this study is to derive a mathematical theory associated with the thermohaline circulations (THC). The results derived provides a general transition and stability theory for the Boussinesq system, governing the motion and states of the large-scale ocean circulation. First, it is shown that the first transition is either to multiple steady states or to oscillations (periodic solutions), determined by the sign of a nondimensional parameter K, depending on the geometry of the physical domain and the thermal and saline Rayleigh numbers. Second, for both the multiple equilibria and periodic solutions transitions, both Type-I (continuous) and Type-II (jump) transitions can occur, and precise criteria are derived in terms of two computable nondimensional parameters b1 and b2. Third, a convection scale law is introduced, leading to an introduction of proper friction terms in the model in order to derive the correct circulation length scale. In particular, the dynamic transitions of the model with the derived friction terms suggest that the THC favors the continuous transitions to stable multiple equilibria. This is joint with Tian Ma

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**Navier-Stokes equations on the beta-plane and the sphere**

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We prove that the solution of the Navier-Stokes equations on the beta-plane and a rotating sphere tends towards zonal flows as the rotation rate tends to infinity. The implications for the dimension of the global attractor is also discussed.

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